On Subordination and Superordination of the Multiplier Transformation for Meromorphic Functions

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Abstract. Using the methods of differential subordination and superordination, sufficient conditions are determined on the multiplier transformation for meromorphic functions in the punctured unit disk to obtain respectively the best dominant and best subordinant. New differential sandwich-type results are also obtained for this multiplier transformation.

1. Motivation and preliminaries

Let Σ_p denote the class of all p-valent functions of the form

$$f(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in U^* := \{z \in \mathbb{C} : 0 < |z| < 1\}, \ p \in \mathbb{N} := \{1, 2, \cdots\}).$$

Motivated by the investigation of the multiplier transformation on the class of univalent meromorphic functions [11, 12, 15, 16, 22, 25, 29, 31, 32], we define the *multiplier transformation* $I_p(n, \lambda)$ on the class Σ_p of meromorphic functions by the infinite series

$$I_p(n, \lambda)f(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda-p}\right)^n a_k z^k \quad (\lambda > p).$$

From the definition, it is clear that the operator $I_p(n, \lambda)$ satisfies the identity

$$(1.1) \qquad z[I_p(n, \lambda)f(z)]' = (\lambda - p)I_p(n+1, \lambda)f(z) - \lambda I_p(n, \lambda)f(z).$$

This identity plays a critical role in obtaining information about functions defined by use of the multiplier transformation. Our results on subordination and superordination in this paper will rely heavily on the identity.

Let $\mathcal{H}(U)$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, with $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let f and F be members of $\mathcal{H}(U)$. The function f is said to be *subordinate* to F, or F is *superordinate* to f, written $f(z) \prec F(z)$, if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = F(w(z))$. If F is univalent, then $f(z) \prec F(z)$ is equivalent to the conditions $f(0) = F(0)$ and $f(U) \subset F(U)$. The well-known fact that a convex function is a starlike function can be seen to be equivalent to the following differential implication for an analytic normalized function p with $p(0) = 1$:

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1 + z}{1 - z} \Rightarrow p(z) \prec \frac{1 + z}{1 - z}.$$

The hypothesis of the above implication is an example of a first-order differential subordination. The general theory on differential subordination and on its numerous applications to univalent function theory can be found in the monograph by Miller and Mocanu [23]. Recently, Miller and Mocanu [24] developed the dual theory of differential superordination, and some of the developments on this subject can be found in the monograph by Bulboacă [8].

For analytic functions defined by means of linear operators, general subordination problems were investigated earlier by Ali *et al.* [3–5], Aghalary *et al.* [1], Aouf and Hosssen [7], and Kim and Srivastava [20] by determining the appropriate classes of admissible functions. Subordination properties of meromorphic functions were investigated in [2,9,10,13,14,17–19,26–28,30,33–36]. In particular, Liu and Owa [21] investigated a subordination problem for meromorphic functions defined by a linear operator D^n; in fact, they determined a class of admissible functions so that

$$\left| h\left(\frac{D^n f(z)}{D^{n-1} f(z)}, \frac{D^{n+1} f(z)}{D^n f(z)}, \frac{D^{n+2} f(z)}{D^{n+1} f(z)} \right) \right| < 1 \Rightarrow \left| \frac{D^n f(z)}{D^{n-1} f(z)} \right| < 1.$$

The aim of this paper is to investigate similar implications in terms of subordination and superordination of functions associated with the multiplier transformation $I_p(n, \lambda)$. The next two sections are devoted to applying the differential subordination and superordination results of Miller and Mocanu [23, Theorem 2.3b, p. 28] and [24, Theorem 1, p. 818] to obtain certain classes of admissible functions that will ensure subordination and superordination implications hold for the multiplier transformation $I_p(n, \lambda)$. Ali *et al.* [6] have considered a similar problem for the Liu-Srivastava linear operator on meromorphic functions. Additionally, several new differential sandwich-type results are obtained.

The following definitions and theorems will be required in our present investigation.

Definition 1.1. [23, Definition 2.2b, p. 21] *Denote by $\mathcal{Q}$ the set consisting of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$ where*

$$E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.2. [23, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and*

$$\operatorname{Re}\left\{ \frac{t}{s} + 1 \right\} \geq k \operatorname{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}(a)$. In this case, we set $\Psi_n[\Omega, q] := \Psi_n[\Omega, M, a]$. When the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.3. [24, Definition 3, p. 817] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z)$, $s = zq'(z)/m$, and*

$$\operatorname{Re}\left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \operatorname{Re}\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

Theorem 1.1. [23, Theorem 2.3b, p. 28] *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

Theorem 1.2. [24, Theorem 1, p. 818] *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and $\psi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U, then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}$$

implies $q(z) \prec p(z)$.

2. Subordination of the multiplier transformation

The following class of admissible functions will be required.

Definition 2.1. *Let Ω be a set in $\mathbb{C}$, and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Theta_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{1}{\lambda - p} \left[(\lambda - p)q(\zeta) + k\zeta q'(\zeta) \right],$$

$$\operatorname{Re}\left\{ \frac{(\lambda - p)(w + u - 2v)}{v - u} \right\} \geq k \operatorname{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

$z \in U,\ \zeta \in \partial U \setminus E(q)\ and\ k \geq 1.$

Theorem 2.1. *Let* $\phi \in \Theta_I[\Omega, q]$. *If* $f \in \Sigma_p$ *satisfies*

$$(2.1) \quad \{\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right) : z \in U\} \subset \Omega,$$

then

$$z^p I_p(n, \lambda)f(z) \prec q(z).$$

Proof. Define the analytic function p in U by

$$(2.2) \qquad\qquad p(z) := z^p I_p(n, \lambda)f(z).$$

In view of the relation (1.1), it follows from (2.2) that

$$(2.3) \qquad (\lambda - p)z^p I_p(n+1, \lambda)f(z) = (\lambda - p)p(z) + zp'(z).$$

Further computations show that

$$(2.4) \quad (\lambda - p)^2 z^p I_p(n+2, \lambda)f(z) = (\lambda - p)^2 p(z) + (2(\lambda - p) + 1)zp'(z) + z^2 p''(z).$$

Now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r, s, t) = r,$$

$$(2.5) \qquad v(r, s, t) = \frac{(\lambda - p)r + s}{\lambda - p},$$

$$w(r, s, t) = \frac{(\lambda - p)^2 r + (2(\lambda - p) + 1)s + t}{(\lambda - p)^2}.$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z)$$

$$(2.6) \qquad = \phi\left(r, \frac{(\lambda - p)r + s}{\lambda - p}, \frac{(\lambda - p)^2 r + (2(\lambda - p) + 1)s + t}{(\lambda - p)^2}; z\right).$$

The proof will make use of Theorem 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.6) that

$$\psi(p(z), zp'(z), z^2 p''(z); z) = \phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z),\right.$$

$$(2.7) \qquad\qquad\qquad \left. z^p I_p(n+2, \lambda)f(z); z\right).$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega.$$

To complete the proof, we next show that the admissibility condition for $\phi \in \Theta_I[h, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda - p)(w + u - 2v)}{v - u},$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.1, $p(z) \prec q(z)$ or equivalently,

$$z^p I_p(n, \lambda)f(z) \prec q(z). \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. In this case, the class $\Theta_I[h(U), q]$ is written as $\Theta_I[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let* $\phi \in \Theta_I[h, q]$ *with* $q(0) = 1$. *If* $f \in \Sigma_p$ *satisfies*

$$(2.8) \qquad \phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n + 1, \lambda)f(z), z^p I_p(n + 2, \lambda)f(z); z\right) \prec h(z),$$

then

$$z^p I_p(n, \lambda)f(z) \prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.1. *Let* $\Omega \subset \mathbb{C}$ *and* q *be univalent in* U *with* $q(0) = 1$. *Let* $\phi \in \Theta_I[\Omega, q_\rho]$ *for some* $\rho \in (0, 1)$ *where* $q_\rho(z) = q(\rho z)$. *If* $f \in \Sigma_p$ *satisfies*

$$\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n + 1, \lambda)f(z), z^p I_p(n + 2, \lambda)f(z); z\right) \in \Omega,$$

then

$$z^p I_p(n, \lambda)f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $z^p I_p(n, \lambda)f(z) \prec q_\rho(z)$. The result now follows from the fact that $q_\rho(z) \prec q(z)$. ∎

Theorem 2.3. *Let* h *and* q *be univalent in* U, *with* $q(0) = 1$, *and set* $q_\rho(z) = q(\rho z)$ *and* $h_\rho(z) = h(\rho z)$. *Let* $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ *satisfy one of the following conditions:*

(1) $\phi \in \Theta_I[h, q_\rho]$ *for some* $\rho \in (0, 1)$, *or*
(2) *there exists* $\rho_0 \in (0, 1)$ *such that* $\phi \in \Theta_I[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ *satisfies (2.8), then*

$$z^p I_p(n, \lambda)f(z) \prec q(z).$$

Proof. The result is similar to the proof in [23, Theorem 2.3d, p. 30] and is therefore omitted. ∎

The next theorem yields the best dominant of the differential subordination (2.8).

Theorem 2.4. *Let* h *be univalent in* U, *and* $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. *Suppose that the differential equation*

$$(2.9) \quad \phi\left(q(z), \; q(z) + \frac{zq'(z)}{\lambda - p}, \; q(z) + \frac{(2(\lambda - p) + 1)zq'(z) + z^2 q''(z)}{(\lambda - p)^2}; z\right) = h(z)$$

has a solution q *with* $q(0) = 1$ *and satisfy one of the following conditions:*

(1) $q \in \mathcal{Q}_1$ *and* $\phi \in \Theta_I[h, q]$,
(2) q *is univalent in* U *and* $\phi \in \Theta_I[h, q_\rho]$ *for some* $\rho \in (0, 1)$, *or*
(3) q *is univalent in* U *and there exists* $\rho_0 \in (0, 1)$ *such that* $\phi \in \Theta_I[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \Sigma_p$ *satisfies (2.8), then*

$$z^p I_p(n, \lambda)f(z) \prec q(z),$$

and q *is the best dominant.*

Proof. Following the same arguments given in [23, Theorem 2.3e, p. 31], we deduce that q is a dominant from Theorems 2.2 and 2.3. Since q satisfies (2.9), it is also a solution of (2.8) and therefore q will be dominated by all dominants. Hence q is the best dominant. ∎

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Theta_I[\Omega, q]$, denoted by $\Theta_I[\Omega, M]$, can be expressed in the following form:

Definition 2.2. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Theta_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that*

$$(2.10) \quad \phi\left(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{kMe^{i\theta}}{\lambda - p}, 1 + Me^{i\theta} + \frac{L + [2(\lambda - p) + 1]kMe^{i\theta}}{(\lambda - p)^2}; z\right) \notin \Omega$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq kM(k - 1)$ for all real θ and $k \geq 1$.

Corollary 2.2. *Let $\phi \in \Theta_I[\Omega, M]$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right) \in \Omega,$$

then

$$z^p I_p(n, \lambda)f(z) \prec 1 + Mz.$$

When $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$. Corollary 2.2 can now be written in the following form:

Corollary 2.3. *Let $\phi \in \Theta_I[M]$. If $f \in \Sigma_p$ satisfies*

$$|\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right) - 1| < M,$$

then

$$|z^p I_p(n, \lambda)f(z) - 1| < M.$$

Example 2.1.

(1) If $f \in \Sigma_p$ satisfies

$$|z^p I_p(n+1, \lambda)f(z) - 1| < M,$$

then

$$|z^p I_p(n, \lambda)f(z) - 1| < M.$$

This implication follows from Corollary 2.3 by taking $\phi(u, v, w; z) = v$.

(2) If $f \in \Sigma_p$, $z^p I_p(n, \lambda)f \in \mathcal{H}$, then

$$|z^p I_p(n+1, \lambda)f(z) - z^p I_p(n, \lambda)f(z)| < \frac{M}{\lambda - p}$$

$$(2.11) \qquad \Rightarrow |z^p I_p(n, \lambda)f(z) - 1| < M.$$

In this case, let $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ with $h(z) = \frac{M}{\lambda - p}z$, $M > 0$. To apply Corollary 2.2, we need to show that $\phi \in \Theta_I[\Omega, M]$, that is, the admissibility condition (2.10) is satisfied. That this hold follows from

$$\left| \phi\left(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{k}{\lambda - p}Me^{i\theta}, 1 + Me^{i\theta} \right.\right.$$
$$\left.\left. + \frac{L + [2(\lambda - p) + 1]kMe^{i\theta}}{(\lambda - p)^2}; z\right) \right| = \frac{k}{\lambda - p}M \geq \frac{M}{\lambda - p}$$

for $z \in U$, $\theta \in \mathbb{R}$ and $k \geq 1$.

Theorem 2.4 shows that the result is sharp. The equation

$$\frac{zp'(z)}{\lambda - p} = z^p I_p(n+1, \lambda)f(z) - z^p I_p(n, \lambda)f(z) = \frac{Mz}{\lambda - p} \quad (\lambda - p < M)$$

has a univalent solution $q(z) = z^p I_p(n, \lambda) f(z) = 1 + Mz$. It follows from Theorem 2.4 that $q(z) = 1 + Mz$ is the best dominant of (2.11).

(3) Let $M > 0$, and $z^p I_p(n, \lambda) f(z) \in \mathcal{H}$. If $f \in \Sigma_p$ satisfies

$$z^p I_p(n + 2, \lambda) f(z) - z^p I_p(n + 1, \lambda) f(z) \prec \frac{M(\lambda - p + 1)z}{(\lambda - p)^2},$$

then

$$|z^p I_p(n, \lambda) f(z) - 1| < M.$$

This implication follows from Corollary 2.2 by taking $\phi(u, v, w; z) = w - v$ and $\Omega = h(U)$ with

$$h(z) = \frac{M(\lambda - p + 1)}{(\lambda - p)^2} z, \quad M > 0.$$

Definition 2.3. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Theta_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \ v = \frac{1}{\lambda - p}\left((\lambda - p)q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right) \quad (q(\zeta) \neq 0),$$

$$\operatorname{Re}\left\{\frac{(\lambda - p)v(w - v)}{v - u} - (\lambda - p)(2u - v)\right\} \geq k\operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U, \ \zeta \in \partial U \setminus E(q)$ *and $k \geq 1$.*

Theorem 2.5. *Let $\phi \in \Theta_{I,1}[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

$$(2.12) \quad \left\{\phi\left(\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)}, \frac{I_p(n + 3, \lambda)f(z)}{I_p(n + 2, \lambda)f(z)}; z\right) : z \in U\right\} \subset \Omega,$$

then

$$\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec q(z).$$

Proof. Define the analytic function p in U by

$$(2.13) \qquad p(z) := \frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)}.$$

Using (1.1) and (2.13) yield

$$(2.14) \qquad \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} = \frac{1}{\lambda - p}\left[(\lambda - p)p(z) + \frac{zp'(z)}{p(z)}\right].$$

Further computations show that

$$\frac{I_p(n + 3, \lambda)f(z)}{I_p(n + 2, \lambda)f(z)} = p(z) + \frac{1}{\lambda - p}\left[\frac{zp'(z)}{p(z)}\right.$$

$$(2.15) \qquad \left. + \frac{(\lambda - p)zp'(z) + \frac{zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{z^2 p''(z)}{p(z)}}{(\lambda - p)p(z) + \frac{zp'(z)}{p(z)}}\right].$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$(2.16) \quad u = r, \quad v = r + \frac{1}{\lambda - p}\left(\frac{s}{r}\right), \quad w = r + \frac{1}{\lambda - p}\left[\frac{s}{r} + \frac{(\lambda - p)s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{(\lambda - p)r + \frac{s}{r}}\right],$$

and let

$$\psi(r, s, t; z) = \phi(u, v, w; z)$$

$$(2.17) \qquad = \phi\left(r, r + \frac{1}{\lambda - p}\left(\frac{s}{r}\right), \ r + \frac{1}{\lambda - p}\left[\frac{s}{r} + \frac{(\lambda - p)s + \frac{s}{r} - (\frac{s}{r})^2 + \frac{t}{r}}{(\lambda - p)r + \frac{s}{r}}\right]; z\right).$$

Using equations (2.13), (2.14) and (2.15), it follows from (2.17) that

$$\psi(p(z), zp'(z), z^2 p''(z); z)$$

$$(2.18) \qquad = \phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right).$$

Hence (2.12) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega.$$

To complete the proof, the admissibility condition for ϕ is next shown to be equivalent to the admissibility condition for ψ as given in Definition 1.2. For this purpose, note that

$$\frac{s}{r} = (\lambda - p)(v - r),$$

$$\frac{t}{r} = (\lambda - p)^2 v(w - v) - \frac{s}{r}\left[(\lambda - p)v + 1 - \frac{2s}{r}\right],$$

$$\text{and} \quad \frac{t}{s} + 1 = (\lambda - p)\left[\frac{v(w - v)}{v - u} - (2u - v)\right].$$

Hence $\psi \in \Psi[\Omega, q]$ and by Theorem 1.1, $p(z) \prec q(z)$ or equivalently,

$$\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec q(z). \qquad \blacksquare$$

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping h of U onto Ω, the class $\Theta_{I,1}[h(U), q]$ is written as $\Theta_{I,1}[h, q]$. Proceeding similarly as before, the following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let $\phi \in \Theta_{I,1}[h, q]$ with $q(0) = 1$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right) \prec h(z),$$

then

$$\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec q(z).$$

In the particular case $q(z) = 1 + Mz$, $M > 0$, the class of admissible functions $\Theta_{I,1}[\Omega, q]$ is simply denoted by $\Theta_{I,1}[\Omega, M]$.

Definition 2.4. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Theta_{I,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that*

$$\phi\Big(1 + Me^{i\theta}, \ 1 + \frac{k + (\lambda - p)(1 + Me^{i\theta})}{(\lambda - p)(1 + Me^{i\theta})}Me^{i\theta}, \ 1 + \frac{k + (\lambda - p)(1 + Me^{i\theta})}{(\lambda - p)(1 + Me^{i\theta})}Me^{i\theta}$$
$$+ \frac{(M + e^{-i\theta})\left[\left((\lambda - p)(1 + Me^{i\theta}) + 1\right)kM + Le^{-i\theta}\right] - k^2M^2}{(\lambda - p)(1 + Me^{i\theta})[(\lambda - p)(M + e^{-i\theta})^2 + kMe^{-i\theta}]}; z\Big) \notin \Omega$$

whenever $z \in U$, $\theta \in \mathbb{R}$, $\mathrm{Re}\left\{Le^{-i\theta}\right\} \geq kM(k - 1)$ for all real θ and $k \geq 1$.

Corollary 2.4. *Let $\phi \in \Theta_{I,1}[\Omega, M]$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)}, \frac{I_p(n + 3, \lambda)f(z)}{I_p(n + 2, \lambda)f(z)}; z\right) \in \Omega,$$

then

$$\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec 1 + Mz.$$

When $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Theta_{I,1}[\Omega, M]$ is simply denoted by $\Theta_{I,1}[M]$.

Corollary 2.5. *Let $\phi \in \Theta_{I,1}[M]$. If $f \in \Sigma_p$ satisfies*

$$\left|\phi\left(\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)}, \frac{I_p(n + 3, \lambda)f(z)}{I_p(n + 2, \lambda)f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} - 1\right| < M.$$

Example 2.2. Let $f \in \Sigma_p$. Then

$$\left|\frac{I_p(n + 2, \lambda)f(z)}{I_p(n + 1, \lambda)f(z)} - \frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)}\right| < \frac{M}{(\lambda - p)(1 + M)}$$
$$\Rightarrow \left|\frac{I_p(n + 1, \lambda)f(z)}{I_p(n, \lambda)f(z)} - 1\right| < M.$$

This implication follows from Corollary 2.4 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ with

$$h(z) = \frac{M}{(\lambda - p)(1 + M)}z.$$

3. Superordination of the multiplier transformation

The dual problem of differential subordination, that is, differential superordination of the multiplier transformation is investigated in this section. For this purpose, the class of admissible functions given in the following definition will be required.

Definition 3.1. *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}$ and $zq'(z) \neq 0$. The class of admissible functions $\Theta'_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \;\; v = \frac{1}{\lambda - p}\left[(\lambda - p)q(z) + \frac{zq'(z)}{m}\right],$$

$$\mathrm{Re}\left\{\frac{(\lambda - p)(w + u - 2v)}{v - u}\right\} \leq \frac{1}{m}\mathrm{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$z \in U, \; \zeta \in \partial U$ *and* $m \geq 1$.

Theorem 3.1. *Let* $\phi \in \Theta'_I[\Omega, q]$. *If* $f \in \Sigma_p$, $z^p I_p(n, \lambda)f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right)$$

is univalent in U, *then*

$$(3.1) \quad \Omega \subset \{\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right) : z \in U\}$$

implies

$$q(z) \prec z^p I_p(n, \lambda)f(z).$$

Proof. Using the transformation (2.5) and the function ψ in (2.6), it follows from (2.7) and (3.1) that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2 p''(z); z\right) : z \in U\right\}.$$

From (2.5), the admissibility condition for $\phi \in \Theta'_I[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec z^p I_p(n, \lambda)f(z). \hspace{3cm}\blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping h of U onto Ω, then the class $\Theta'_I[h(U), q]$ is written as $\Theta'_I[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let* $q \in \mathcal{H}$, h *be analytic in* U *and* $\phi \in \Theta'_I[h, q]$. *If* $f \in \Sigma_p$, $z^p I_p(n, \lambda)f(z) \in \mathcal{Q}_1$ *and* $\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right)$ *is univalent in* U, *then*

$$(3.2) \quad h(z) \prec \phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right)$$

implies

$$q(z) \prec z^p I_p(n, \lambda)f(z).$$

Theorems 3.1 and 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for an appropriate ϕ.

Theorem 3.3. *Let* h *be analytic in* U *and* $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. *Suppose that the differential equation*

$$\phi\left(q(z), \frac{(\lambda - p)q(z) + zq'(z)}{\lambda - p}, \frac{(\lambda - p)^2 q(z) + (2(\lambda - p) + 1)zq'(z) + z^2 q''(z)}{(\lambda - p)^2}; z\right) = h(z)$$

has a solution $q \in \mathcal{Q}_1$. *If* $\phi \in \Theta'_I[h, q]$, $f \in \Sigma_p$, $z^p I_p(n, \lambda)f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(z^p I_p(n, \lambda)f(z), z^p I_p(n+1, \lambda)f(z), z^p I_p(n+2, \lambda)f(z); z\right)$$

is univalent in U, then

$$h(z) \prec \phi \left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z \right)$$

implies

$$q(z) \prec z^p I_p(n, \lambda) f(z),$$

and q is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.4 and is omitted. ∎

Theorems 2.2 and 3.2 can be combined to obtain the following differential sandwich-type theorem.

Corollary 3.1. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent in U, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$, and $\phi \in \Theta_I[h_2, q_2] \cap \Theta'_I[h_1, q_1]$. If $f \in \Sigma_p$, $z^p I_p(n, \lambda) f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi \left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z \right)$$

is univalent in U, then

$$h_1(z) \prec \phi \left(z^p I_p(n, \lambda) f(z), z^p I_p(n+1, \lambda) f(z), z^p I_p(n+2, \lambda) f(z); z \right) \prec h_2(z),$$

implies

$$q_1(z) \prec z^p I_p(n, \lambda) f(z) \prec q_2(z).$$

Definition 3.2. *Let Ω be a set in $\mathbb{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ and $q \in \mathcal{H}$. The class of admissible functions $\Theta'_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{1}{\lambda - p} \left((\lambda - p) q(z) + \frac{zq'(z)}{mq(z)} \right),$$

$$\text{Re} \left\{ \frac{(\lambda - p) v(w - v)}{v - u} - (\lambda - p)(2u - v) \right\} \leq \frac{1}{m} \text{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$, $\zeta \in \partial U$ and $m \geq 1$.

We now give the dual result of Theorem 2.5 for differential superordination.

Theorem 3.4. *Let $\phi \in \Theta'_{I,1}[\Omega, q]$ and $q \in \mathcal{H}$. If $f \in \Sigma_p$, $\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)} \in \mathcal{Q}_1$, and*

$$\phi \left(\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)}, \frac{I_p(n+2, \lambda) f(z)}{I_p(n+1, \lambda) f(z)}, \frac{I_p(n+3, \lambda) f(z)}{I_p(n+2, \lambda) f(z)}; z \right)$$

is univalent in U, then

$$(3.3) \quad \Omega \subset \left\{ \phi \left(\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)}, \frac{I_p(n+2, \lambda) f(z)}{I_p(n+1, \lambda) f(z)}, \frac{I_p(n+3, \lambda) f(z)}{I_p(n+2, \lambda) f(z)}; z \right) : z \in U \right\}$$

implies

$$q(z) \prec \frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)}.$$

Proof. The relations (2.18) and (3.3) yield

$$\Omega \subset \left\{ \phi\left(p(z), zp'(z), z^2 p''(z); z\right) : z \in U \right\}.$$

From (2.16), the admissibility condition for ϕ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}. \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h of U onto Ω. In this case, the class $\Theta'_{I,1}[h(U), q]$ is written as $\Theta'_{I,1}[h, q]$. The following result is an immediate consequence of Theorem 3.4.

Theorem 3.5. *Let $q \in \mathcal{H}$, h be analytic in U and $\phi \in \Theta'_{I,1}[h, q]$. If $f \in \Sigma_p$, $\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \in \mathcal{Q}_1$, and $\phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right)$ is univalent in U, then*

$$h(z) \prec \phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right)$$

implies

$$q(z) \prec \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}.$$

Theorems 2.6 and 3.5 taken together yield the following differential sandwich theorem.

Corollary 3.2. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent in U, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$, and $\phi \in \Theta_{I,1}[h_2, q_2] \cap \Theta'_{I,1}[h_1, q_1]$. If $f \in \Sigma_p$, $\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$, and $\phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right)$ is univalent in U, then*

$$h_1(z) \prec \phi\left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)}, \frac{I_p(n+2, \lambda)f(z)}{I_p(n+1, \lambda)f(z)}, \frac{I_p(n+3, \lambda)f(z)}{I_p(n+2, \lambda)f(z)}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} \prec q_2(z).$$

References

[1] R. Aghalary, R. M. Ali, S. B. Joshi and V. Ravichandran, Inequalities for analytic functions defined by certain linear operators, *Int. J. Math. Sci.* **4** (2005), no. 2, 267–274.

[2] R. M. Ali and V. Ravichandran, Differential subordination for meromorphic functions defined by a linear operator, *J. Anal. Appl.* **2** (2004), no. 3, 149–158.

[3] R. M. Ali, V. Ravichandran and N. Seenivasagan, Differential subordination and superordination of analytic functions defined by the Dziok-Srivastava linear operator, *preprint*.

[4] R. M. Ali, V. Ravichandran and N. Seenivasagan, Differential subordination and superordination of analytic functions defined by the multiplier transformation, *Math. Inequal. Appl.* **12** (2009), no. 1, 123–139.

[5] R. M. Ali, V. Ravichandran and N. Seenivasagan, Subordination and superordination on Schwarzian derivatives, *J. Inequal. Appl.* **2008**, Art. ID 712328, 18 pp.

[6] R. M. Ali, V. Ravichandran and N. Seenivasagan, Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions, *Bull. Malays. Math. Sci. Soc.* (*2*) **31** (2008), no. 2, 193–207.

[7] M. K. Aouf and H. M. Hossen, New criteria for meromorphic p-valent starlike functions, *Tsukuba J. Math.* **17** (1993), no. 2, 481–486.

[8] T. Bulboacă, *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[9] N. E. Cho, On certain classes of meromorphically multivalent functions, *Math. Japon.* **40** (1994), no. 3, 497–501.

[10] N. E. Cho, On certain subclasses of meromorphically multivalent functions, *Tamkang J. Math.* **26** (1995), no. 3, 251–255.

[11] N. E. Cho and S. H. Lee, Certain subclasses of meromorphic functions defined by subordination. II, *Kyungpook Math. J.* **36** (1996), no. 2, 283–291.

[12] N. E. Cho and I. H. Kim, Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.* **187** (2007), no. 1, 115–121.

[13] N. E. Cho and J. A. Kim, On certain classes of meromorphically starlike functions, *Internat. J. Math. Math. Sci.* **18** (1995), no. 3, 463–467.

[14] N. E. Cho and O. S. Kwon, A class of integral operators preserving subordination and superordination, *Bull. Malays. Math. Sci. Soc.* (*2*), to appear.

[15] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations, *Integral Transforms Spec. Funct.* **16** (2005), no. 8, 647–659.

[16] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations, *J. Math. Anal. Appl.* **300** (2004), no. 2, 505–520.

[17] N. E. Cho and K. I. Noor, Inclusion properties for certain classes of meromorphic functions associated with the Choi-Saigo-Srivastava operator, *J. Math. Anal. Appl.* **320** (2006), no. 2, 779–786.

[18] E. Drăghici, About an integral operator preserving meromorphic starlike functions, *Bull. Belg. Math. Soc. Simon Stevin* **4** (1997), no. 2, 245–250.

[19] H. Irmak, Some applications of Hadamard convolution to multivalently analytic and multivalently meromorphic functions, *Appl. Math. Comput.* **187** (2007), no. 1, 207–214.

[20] Y. C. Kim and H. M. Srivastava, Inequalities involving certain families of integral and convolution operators, *Math. Inequal. Appl.* **7** (2004), no. 2, 227–234.

[21] J. Liu and S. Owa, On certain meromorphic p-valent functions, *Taiwanese J. Math.* **2** (1998), no. 1, 107–110.

[22] J.-L. Liu, A linear operator and its applications on meromorphic p-valent functions, *Bull. Inst. Math. Acad. Sinica* **31** (2003), no. 1, 23–32.

[23] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Dekker, New York, 2000.

[24] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.* **48** (2003), no. 10, 815–826.

[25] J. Patel and P. Sahoo, On certain subclasses of meromorphically p-valent functions, *Bull. Calcutta Math. Soc.* **93** (2001), no. 6, 455–464.

[26] K. Piejko and J. Sokół, Subclasses of meromorphic functions associated with the Cho-Kwon-Srivastava operator, *J. Math. Anal. Appl.* **337** (2008), no. 2, 1261–1266.

[27] V. Ravichandran, S. Sivaprasad Kumar and K. G. Subramanian, Convolution conditions for spirallikeness and convex spirallikeness of certain meromorphic p-valent functions, *JIPAM. J. Inequal. Pure Appl. Math.* **5** (2004), no. 1, Article 11, 7 pp. (electronic).

[28] V. Ravichandran, S. Sivaprasad Kumar and M. Darus, On a subordination theorem for a class of meromorphic functions, *JIPAM. J. Inequal. Pure Appl. Math.* **5** (2004), no. 1, Article 8, 4 pp. (electronic).

[29] S. M. Sarangi and S. B. Uralegaddi, Certain differential operators for meromorphic functions, *Bull. Calcutta Math. Soc.* **88** (1996), no. 4, 333–336.

[30] S. Supramaniam, R. M. Ali, S. K. Lee and V. Ravichandran, Convolution and differential subordination for multivalent functions, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 351–360.

[31] B. A. Uralegaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions, *Bull. Austral. Math. Soc.* **43** (1991), no. 1, 137–140.

[32] B. A. Uralegaddi and C. Somanatha, Certain differential operators for meromorphic functions, *Houston J. Math.* **17** (1991), no. 2, 279–284.

[33] R.-G. Xiang, Z.-G. Wang and M. Darus, A family of integral operators preserving subordination and superordination, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 1, 121–131.

[34] N. Xu and D. Yang, On starlikeness and close-to-convexity of certain meromorphic functions, *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **10** (2003), no. 1, 1–11.

[35] D. Yang, On a class of meromorphic starlike multivalent functions, *Bull. Inst. Math. Acad. Sinica* **24** (1996), no. 2, 151–157.

[36] S.-M. Yuan, Z.-M. Liu and H. M. Srivastava, Some inclusion relationships and integral-preserving properties of certain subclasses of meromorphic functions associated with a family of integral operators, *J. Math. Anal. Appl.* **337** (2008), no. 1, 505–515.